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The deformation matrix and the deformation ellipsoid

DEREK FLINN

Jane Herdman Laboratories of Geology, University of Liverpool, Brownlow Street, PO Box 147, Liverpool L69 3BX, England

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Abstract—Homogeneous strain can be computed most easily by the methods of matrix algebra. Lines, planes and ellipsoids represented in matrix form can be homogeneously deformed by simple matrix multiplication by linear transformation matrices, the elements of which are the coefficients of the transformation equations. Deformation matrices or linear transformation matrices which cause geological-type homogeneous strain are divided into four classes based on the presence or absence of symmetry and/or orthogonality. The nature of the homogeneous strain caused by each class of deformation matrix is examined. Orthogonal-symmetrical and orthogonal matrices cause rotation. Symmetrical matrices cause irrotational strain with co-axial strain as a special case. Matrices which are neither orthogonal nor symmetrical cause many different types of rotational strain, some of which are examined.

INTRODUCTION

IN STRUCTURAL geology text-books the study of strain is often introduced by citing the linear transformation equations:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \qquad (1)$$

and then deriving from them a series of algebraic expressions defining the geometry of deformation (e.g. Jaeger 1956). Presentation is often confined to two-dimensional deformation because the algebraic equations become very cumbersome for three-dimensional deformation.

The deformation and its geometry are much more easily studied when represented in matrix form. The matrix algebra required may be found in elementary text-books but not presented in terms of three-dimensional deformation and lacking the necessary co-ordinate geometry.

Equations (1) are more conveniently represented in matrix form as follows:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad (2)$$

where $\mathbf{x} = (x_1 x_2 x_3)^T$

and $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$\mathbf{A}$ is the linear transformation matrix when used in the relation (2) because it is then an operator performing linear transformation.

For geological-type deformation the elements of $\mathbf{A}$ must all be real and $\mathbf{A}$ must be non-singular, that is det $\mathbf{A} \neq 0$. The necessity for real elements for geological-type operations is obvious. If det $\mathbf{A} = 0$ then the transformation becomes a projection or mapping of three dimensions onto two- or even one-dimensional space and no geological deformation can take this form. In the rest of this paper $\mathbf{D}$ will be a 3×3 real, non-singular linear transformation matrix and it will be called the deformation matrix. However, it will be shown that not all such matrices give rise to geological-type deformation.

HOMOGENEOUS STRAIN AND THE DEFORMATION MATRIX

In (2) if a given $\mathbf{D}$ is substituted for $\mathbf{A}$ and $\mathbf{x}$ is set equal in turn to all unit vectors then $\mathbf{x}'$ becomes in turn all radius vectors of the form into which $\mathbf{D}$ transforms a sphere of unit radius.

In matrix terms this transformation is performed as follows. Equation (2) is solved for $\mathbf{x}$ in terms of $\mathbf{x}'$:

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{x}'$$

and substituted into the unit sphere

$$x_1^2 + x_2^2 + x_3^2 = 1 \qquad (3)$$

represented by the identity matrix, $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

as follows: $\quad (\mathbf{D}^{-1})^T \mathbf{I} \mathbf{D}^{-1} \qquad (4)$

simplifying to $\quad (\mathbf{D}^{-1})^T \mathbf{D}^{-1} = \mathbf{Q}, \qquad (5)$

$\mathbf{Q} = \begin{bmatrix} r & u & v \\ u & s & w \\ v & w & t \end{bmatrix}$ being the matrix representation of the central quadric

$$r x_1^2 + s x_2^2 + t x_3^2 + 2 u x_1 x_2 + 2 v x_1 x_3 + 2w x_2 x_3 = 1. \qquad (6)$$

The same transformation can be performed algebraically. Equations (1) are solved for $\mathbf{x}$ in terms of $\mathbf{x}'$ giving:

$$\begin{aligned} x_1 &= a_{11}'x_1' + a_{12}'x_2' + a_{13}'z_3' \\ x_2 &= a_{21}'x_1 + a_{22}'x_2' + a_{23}'z_3' \\ x_3 &= a_{31}'x_1' + a_{32}'x_2' + a_{33}'x_3' \end{aligned} \qquad (7)$$

where a'_{ij} are linear functions of a_{ij}. Equations (7) are substituted in equation (3) to give:

$$(a_{11}'x_1' + a_{12}'x_2' + a_{13}'x_3')^2 + (a_{21}'x_1' + a_{22}'x_2' + a_{23}'x_3')^2 + (a_{31}'x_1' + a_{32}'x_2' + a_{33}x_3')^2 = 1. \qquad (8)$$

Equation (8) can be expanded and simplified to the form (6) obtained before

$$rx_1^2 + sx_2^2 + tx_3^2 + 2ux_1x_2 + 2vx_1x_3 + 2wx_2x_3 = 1$$

where $r\ s - - w$ are all simple but lengthy function of a_{ij} and therefore of a_{ij}.

Equation (6) is the standard form for all central quadrics centred on the origin and is most conveniently represented by the matrix $\boldsymbol{Q}$. The particular form of a central quadric is usually determined from its eigenvalues. Here, a quicker method is available. The LHS of equation (8) is a sum of squares and is therefore positive for all real non-zero values of x_1, x_2 and x_3. Thus, $\boldsymbol{Q}$ in (5) is positive definite (it has three non-zero positive eigenvalues) and therefore both $\boldsymbol{Q}$ and equation (6) represent an ellipsoid centred on the origin of the reference axes and completely specified by the elements of $\boldsymbol{Q}$ with respect to shape, volume and orientation relative to the reference axes, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. In the rest of the paper such positive definite matrices will be called $\boldsymbol{E}$.

In (4) above, a sphere, $\boldsymbol{I}$, was deformed to an ellipsoid. Any ellipsoid of any orientation may be deformed by substituting its matrix, $\boldsymbol{E}_0$, for $\boldsymbol{I}$ in (4)

$$(\boldsymbol{D}^{-1})^T\boldsymbol{E}^0\boldsymbol{D}^{-1} = \boldsymbol{Q}_R. \tag{9}$$

Again, $\boldsymbol{Q}_R$ is a central quadric whose form, normally is most easily obtained from the eigenvalues. Here the determination may be done as before. It is clear that substitution of equations (7) into a sum of squares equation (8) representing any ellipsoid leaves the sum of squares intact so that $\boldsymbol{Q}_R$ in (9) is again positive definite and represents an ellipsoid. Thus

$$(\boldsymbol{D}^{-1})^T\boldsymbol{E}_0\boldsymbol{D}^{-1} = \boldsymbol{E}_R. \tag{9a}$$

In geological terms $\boldsymbol{E}_0$ is the starting deformation ellipsoid, and $\boldsymbol{E}_R$ is the resultant deformation ellipsoid. The deformation ellipsoid, $\boldsymbol{E} = (\boldsymbol{D}^{-1})^T(\boldsymbol{D}^{-1}) = \boldsymbol{Q}$ from (5), representing the deformation caused by $\boldsymbol{D}$, is the incremental (or the superposed) deformation ellipsoid.

The deformation matrix may be used to deform planes and lines as well as ellipsoids. The equation of a plane passing through the origin of the reference axes is given by

$$lx_1 + mx_2 + nx_3 = 0 \tag{10}$$

where l, m and n are the direction cosines of the normal of the plane.

In algebraic terms the plane may be deformed by the substitution of (7) into (10) to give:

$l(a'_{11}x'_1 + a'_{12}x'_2 + a'_{13}x'_3) + m(a'_{21}x'_1 + a'_{22}x'_2 + a'_{23}x'_3) + n(a'_{31}x'_1 + a'_{32}x'_2 + a'_{33}x'_3) = 0$

which simplifies to the standard form for a plane

$(la'_{11} + ma'_{21} + na'_{31})\,x'_1 + (la'_{12} + ma'_{22} + na'_{32})\,x'_2 + (la'_{13} + ma'_{23} + na'_{33})\,x'_3 = 0.$

Therefore, during deformations by $\boldsymbol{D}$, planes remain planes and, furthermore, parallel planes remain parallel. In matrix terms this deformation takes the form

$$\boldsymbol{p}_1^T = \boldsymbol{p}_0^T\boldsymbol{D}^{-1} \tag{11}$$

where $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ are the direction cosines and the direction numbers respectively of the plane normal before and after the deformation.

Straight lines are defined by the intersection of non-parallel planes, therefore, during deformation by $\boldsymbol{D}$, straight lines remain straight lines and parallel lines remain parallel. In matrix terms the deformation of straight lines is given by

$$\boldsymbol{l}_1 = \boldsymbol{D}\,\boldsymbol{l}_0 \tag{12}$$

where $\boldsymbol{l}_0$ and $\boldsymbol{l}_1$ are the direction cosines and direction numbers respectively of the line before and after the deformation. The factor by which the line has been elongated, e, is given by

$$e^2 = \boldsymbol{l}_1^T\,\boldsymbol{l}_1. \tag{13}$$

Thus, the operation performed by the linear transformation matrix or deformation matrix, $\boldsymbol{D}$, is homogeneous strain. However, there are many types of homogeneous strain which are not geologically possible even though they obey the usual rules that straight lines remain straight. It is the main purpose of this paper to identify and examine those deformation matrices which produce deformations which might occur geologically.

Equations (11) and (12) can be used to compute a deformation in a single step, but if written in the form

$$\boldsymbol{l}_{i+1} = \boldsymbol{D}\boldsymbol{l}_i \qquad \boldsymbol{p}_{i+1}^T = \boldsymbol{p}_i^T\boldsymbol{D}^{-1}$$

can be used to compute a progressive deformation in a series of steps or increments. The successive solutions can then be used to plot the movement paths of the poles of the planes and lineations. Equation (9a) in the form

$$(\boldsymbol{D}_i^{-1})^T\,\boldsymbol{E}_i\,\boldsymbol{D}_i^{-1} = \boldsymbol{E}_{i+1}$$

can be used to compute successive resultant deformation ellipsoids and thus plot the deformation path.

An alternative approach is to construct a deformation matrix of deformation rates involving a factor t representing time, allowing the deformation to be run for any desired period of time (Ramberg & Ghosh 1977). This method is limited to the computation of progressive deformations with deformation paths of constant k-value; the incremental deformation ellipsoid is invariant.

Before giving further consideration to the deformation matrix, it is necessary to define some terms. For each deformation matrix, $\boldsymbol{D}$, there is a deformation ellipsoid, $\boldsymbol{E}$, given by

$$\boldsymbol{E} = (\boldsymbol{D}^{-1})^T\,\boldsymbol{D}^{-1} \qquad \text{from (5)}$$

where

$$\boldsymbol{E} = \boldsymbol{R}\,\boldsymbol{M}\,\boldsymbol{R}^T$$

$$\left[\text{and } \boldsymbol{M} = \begin{bmatrix} 1/X_1^2 & 0 & 0 \\ 0 & 1/X_2^2 & 0 \\ 0 & 0 & 1/X_3^2 \end{bmatrix}\right]$$

Note that $1/X_1^2$, $1/X_2^2$ and $1/X_3^2$ are the eigenvalues of $\boldsymbol{E}$, and that $\boldsymbol{R}$ is a 3×3 orthogonal matrix the columns of which are the eigenvectors of $\boldsymbol{E}$ in the same order as the eigenvalues. Also, X_1, X_2, and X_3 give the magnitudes of the three principal axes of the ellipsoid but not necessarily in any given order of magnitude, and the eigenvectors are the direction cosines defining the orientations of the respective axes. Alternatively, the principal axes of deformation ellipsoid may be called the major, intermediate and minor axes and defined as follows:

$$X > Y > Z \text{ and } XYZ = V/V_0 = d$$

where V_0 and V are the volumes of the deformation ellipsoid before and after deformation by $\boldsymbol{D}$. Thus, d is the dilatation factor. If the original sphere is defined as having unit volume then d is the volume of the deformation ellipsoid as well as its dilatation factor. However, it is usually more convenient to define the original sphere as having unit radius, in which case the volume of the ellipsoid is given by

$$V = \tfrac{4}{3}\pi d = \tfrac{4}{3}\pi(\det \boldsymbol{E})^{-1/2} = \tfrac{4}{3}\pi \det \boldsymbol{D}.$$

These relations depend on

$$\det \boldsymbol{E} = 1/(x^2y^2z^2) \;\&\; \det \boldsymbol{E} = \det((\boldsymbol{D}^{-1})^T\boldsymbol{D}^{-1}) = 1/(\det \boldsymbol{D})^2.$$

In this paper the original sphere is defined as having unit radius.

Although every deformation matrix, $\boldsymbol{D}$, has associated with it one particular deformation ellipsoid $\boldsymbol{E}$, given by (13), nevertheless that deformation matrix is not unique to that deformation ellipsoid. For this reason any classification of deformations must be based on the deformation matrix and not on either the deformation ellipsoid or on any or all of its parameters.

Deformation matrices may be divided into the following four main classes based on the fact that such matrices may be either symmetrical or nonsymmetrical, and independently of this they may be orthogonal or non-orthogonal: orthogonal–symmetrical, orthogonal–non-symmetrical, symmetrical–non-orthogonal and non-orthogonal–non-symmetrical. It should be noted that a rigid body rotation of $\boldsymbol{D}$ relative to the reference axes, that is an orthogonal similarity transform of the type

$$\boldsymbol{D}' = \boldsymbol{R}\,\boldsymbol{D}\,\boldsymbol{R}^T$$

where $\boldsymbol{R}$ is any orthogonal matrix, can change neither the state of symmetry nor the state of orthogonality of $\boldsymbol{D}$. Therefore, this classification of deformation matrices is invariant under a change of base or a rigid body rotation. In the above cited relationship $\boldsymbol{D}'$ is said to be *similar* to $\boldsymbol{D}$. The word 'similar' wherever used below, has this meaning.

The systems of co-ordinate axes, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ employed by mathematicians are usually, but not invariably, right-handed. However, the geological system involving a clockwise graduated compass rose and a positive zenith direction is left-handed. Since the analytical geometry used in this paper may be expressed in either right- or left-handed terms the latter has been employed as being more geological.

ORTHOGONAL–SYMMETRICAL DEFORMATION MATRICES, ${}^{os}\boldsymbol{D}$

Symmetrical matrices have the form

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$

Orthogonal matrices are those in which the sum of squares of the three elements in each column and in each row are equal to unity. Each row and each column may therefore be treated as direction cosines. The three rows form a mutually orthogonal set of directions as do the three columns. Most of the properties of orthogonal matrices are discussed more conveniently in the next section. This section is concerned with orthogonal matrices which are also symmetrical. For orthogonal-symmetrical matrices

$${}^{os}\boldsymbol{D}^{-1} = {}^{os}\boldsymbol{D}^T = {}^{os}\boldsymbol{D}$$

therefore (5) simplifies to

$$({}^{os}\boldsymbol{D}^{-1})^T\,{}^{os}\boldsymbol{D}^{-1} = {}^{os}\boldsymbol{D}^2 = \boldsymbol{E} = \boldsymbol{RMR}^T = \boldsymbol{I}$$

so that

$${}^{os}\boldsymbol{D} = \boldsymbol{E}^{1/2} = \boldsymbol{R}\boldsymbol{M}^{1/2}\boldsymbol{R}^T = \boldsymbol{I}^{1/2}\,. \qquad (14)$$

$\boldsymbol{M}^{1/2}$ is a diagonal matrix and since it is similar to the orthogonal matrix ${}^{os}\boldsymbol{D}$ it must also be orthogonal. Thus, the elements along the leading diagonal must be ± 1. If the special case of $\boldsymbol{R} = \boldsymbol{I}$ is considered then there are four different solutions to (14) given by the following sets of elements forming the leading diagonal in unspecified order: $+1, +1, +1; -1, -1, +1; -1, +1, +1; -1, -1, -1$. The first solution is the identity matrix which when serving as a deformation matrix gives rise to zero deformation. The solutions with one positive and two negative unit roots cause rotations of 180° (and no change of length) about the reference axis association with the positive root. The solutions with a single negative root and two positive ones cause a reflection across the plane normal to the reference axis associated with the negative root. It may be noted that in the solution with two negative roots mentioned above two such reflections occur, but two reflections across orthogonal planes are exactly equivalent to a rotation of 180° about the line of intersection of the planes. The solution with three negative roots causes three orthogonal reflections which is equivalent to an inversion across the origin of the reference axes.

The deformation matrices involving 1 or 3 reflections, that is those for which $\det {}^{os}\boldsymbol{D} = -1$, do not cause geological-type deformations. Those for which $\det {}^{os}\boldsymbol{D} = +1$, cause no deformation or a rotation of 180° without change of length since all orthogonal-symmetrical deformation matrices are associated with unit radius spheres $\boldsymbol{I}$ as deformation ellipsoids.

In relation (14) if $\boldsymbol{R}$ is any orthogonal matrix then the same interpretations apply except that the axial directions are no longer parallel to a reference axis but to a direction defined by one of the columns of $\boldsymbol{R}$ treated as direction cosines. Thus, when $\det {}^{os}\boldsymbol{D} = +1$ the eigenvector associated with the single positive root fixes the attitude relative to the reference axes of the axis about which the 180° rotation takes place. A deformation matrix whose operation is 180° rotation about an axis at any desired inclination to the reference axes may be constructed by substituting into (14) a suitably chosen $\boldsymbol{R}$. All such matrices will be both orthogonal and symmetrical.

ORTHOGONAL NON-SYMMETRICAL DEFORMATION MATRICES, ${}^{o}\boldsymbol{D}$

For orthogonal matrices ${}^{o}\boldsymbol{D}^{-1} = {}^{o}\boldsymbol{D}^T$ so that equation (5) simplifies to

$$(\boldsymbol{D}^{-1})^T\,\boldsymbol{D}^{-1} = {}^{o}\boldsymbol{D}\,{}^{o}\boldsymbol{D}^T = \boldsymbol{E} = \boldsymbol{RMR}^T = \boldsymbol{I}. \qquad (15)$$

Taking first the case where $\boldsymbol{R} = \boldsymbol{I}$ so that

$${}^{o}\boldsymbol{D}\,{}^{o}\boldsymbol{D}^T = \boldsymbol{I}^{1/2}\,(\boldsymbol{I}^{1/2})^T$$

there are two solutions, as follows, for every value of θ from 0 to 2π

$${}^{o}\boldsymbol{D} = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}. \qquad (16)$$

The eigenvalues of these two solutions and of all orthogonal non-symmetrical matrices are ± 1, $\cos\theta \pm i\sin\theta$.

Therefore, $\det {}^{o}\boldsymbol{D} = \pm 1$ for all orthogonal matrices. The operation performed by all orthogonal matrices is rotation if the real root is $+1$ and rotation together with reflection across the plane normal to the rotation axis if the real root is -1. Therefore, only deformation matrices with $\det {}^{o}\boldsymbol{D} = +1$ cause geological-type deformation. For (15) it is clear that the deformation ellipsoid is a sphere of unit radius so that no change of length accompanies the rotations.

The operation performed by ${}^{o}\boldsymbol{D}$ [(16) and above] when $\det {}^{o}\boldsymbol{D} = +1$ is a rotation of $\theta°$ clockwise about the $\mathbf{x}_1$ reference axis considered from the positive end of the axis looking towards the origin (Flinn 1979, pp. 90–93, Figs. 2 & 3).

By means of a suitably chosen $\boldsymbol{R}$ in

$${}^{o}\boldsymbol{D}' = \boldsymbol{R}\,{}^{o}\boldsymbol{D}\boldsymbol{R}^T$$

the rotation axis may be given any desired orientation because the rotation caused by ${}^{o}\boldsymbol{D}'$ takes place about the direction defined by the column of $\boldsymbol{R}$ associated with the real unit root [the first column for ${}^{o}\boldsymbol{D}$ as in (16)], and the rotation is $\theta°$ where the complex roots are $\cos\theta \pm i\sin\theta$. Alternatively the direction cosines of the rotation axis are

$$(l\ m\ n)^T = (a_{23}-a_{32}\ \ a_{31}-a_{13}\ \ a_{12}-a_{21})^T$$

$$\text{if } {}^{o}\boldsymbol{D} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and the angle of rotation θ is

$$\cos\theta = \tfrac{1}{2}(\mathrm{tr}\ {}^{o}\boldsymbol{D} - 1) = \tfrac{1}{2}(a_{11} + a_{22} + a_{33} - 1).$$

${}^{o}D$ may be used to rotate directions $\boldsymbol{l}_0$ to $\boldsymbol{l}_1$ by means of (12), that is

$$\boldsymbol{l}_1 = {}^{o}\boldsymbol{D}\,\boldsymbol{l}_0.$$

The columns of ${}^{o}D$ are the new direction cosines of the directions which before the deformation were parallel to the reference axes, while the rows of ${}^{o}\boldsymbol{D}$ are the direction cosines before the deformation, of directions which are parallel to the reference axis after the deformation. The operation is a rigid body rotation since the angle between any two directions is the same before and after the rotation.

All orthogonal deformation matrices with positive unit determinant may be used to perform rigid body rotations on other deformation matrices, orthogonal or otherwise, and on ellipsoids by means of orthogonal similarity transformations as follows:

$$\boldsymbol{D}_R = {}^{o}\boldsymbol{D}_1\boldsymbol{D}_0\,{}^{o}\boldsymbol{D}_1^T$$

$$\boldsymbol{E}_R = {}^{o}\boldsymbol{D}_1\boldsymbol{E}_0\,{}^{o}\boldsymbol{D}_1^T.$$

The matrix $\boldsymbol{R}$ used throughout this paper is an orthogonal matrix of type $\det({}^{o}\boldsymbol{D}) = +1$. The symbol $\boldsymbol{R}$ is used instead of ${}^{o}\boldsymbol{D}$ when the matrix is constructed with eigenvectors to be used to set the orientation of another matrix rather than to 'deform' it. There is no mathematical difference, only one of geological interpretation.

A sequence of rotations ${}^{o}\boldsymbol{D}_1$, ${}^{o}\boldsymbol{D}_2$, ${}^{o}\boldsymbol{D}_3$ etc. can be combined as follows:

$${}^{o}\boldsymbol{D}_R = {}^{o}\boldsymbol{D}_3\,{}^{o}\boldsymbol{D}_2\,{}^{o}\boldsymbol{D}_1$$

provided the rotations took place in the order 1, 2, 3. In general, a different resultant matrix, ${}^{o}\boldsymbol{D}_R$, and therefore a different resultant deformation would be obtained if they were multiplied together in a different order. Orthogonal deformation matrices can be multiplied together in any order only when they have parallel rotation axes.

SYMMETRICAL (NON-ORTHOGONAL) MATRICES, ${}^{s}\mathbf{D}$

For symmetrical matrices ${}^{s}\boldsymbol{D} = {}^{s}\boldsymbol{D}^T$ so that equation (5) simplifies to

$$(\boldsymbol{D}^{-1})^T\boldsymbol{D}^{-1} = {}^{s}\boldsymbol{D}^{-1}\,{}^{s}\boldsymbol{D}^{-1} = \boldsymbol{E} = \boldsymbol{R}\boldsymbol{M}\boldsymbol{R}^T \quad (17)$$

where $\boldsymbol{M}$ is a diagonal matrix in which not all the diagonal elements are unity and in most cases none are. It follows that

$${}^{s}\boldsymbol{D} = \boldsymbol{R}\,(\boldsymbol{M}^{1/2})^{-1}\boldsymbol{R}^T \quad (18)$$

$$\text{where } (\boldsymbol{M}^{1/2})^{-1} = \begin{bmatrix} \pm X_1 & 0 & 0 \\ 0 & \pm X_2 & 0 \\ 0 & 0 & \pm X_3 \end{bmatrix}$$

X_1, X_2 and X_3 being the principal axes of the deformation ellipsoid, $\boldsymbol{E}$ in (17). If $\boldsymbol{R} = \boldsymbol{I}$ in (18) then the operation performed by ${}^{s}\boldsymbol{D}$ in (19)

$$\boldsymbol{l}_1 = {}^{s}\boldsymbol{D}\boldsymbol{l}_0 \quad (19)$$

is a change of length of a direction parallel to the $\mathbf{x}$ reference axis by a factor $\pm X_1$, of a direction parallel to the $\mathbf{x}_2$ reference axis by a factor $\pm X_2$, and of a direction parallel to the $\mathbf{x}_3$ reference axis by a factor $\pm X_3$, all without a change of orientation. All other directions suffer a change of length by some intermediate factor together with a change of orientation. A negative factor causes a change of length together with a reflection across the plane normal to the axis associated with the factor. Therefore, only positive definite symmetrical deformation matrices are of geological significance. If $\det {}^{s}\boldsymbol{D}$ is negative there are either one or three negative factors (eigenvalues) and if the determinant is positive but the matrix is not positive definite there are two negative factors (eigenvalues) all associated with reflections.

If ${}^{s}\boldsymbol{D}$ has three positive eigenvalues, that is ${}^{s}\boldsymbol{D}$ is positive definite, then its operation is pure shear or three-dimensional irrotational strain. If $\boldsymbol{R} = \boldsymbol{I}$ in (17) and (18) then the three irrotational directions, the axes of the deformation ellipsoid, are parallel to the reference axes. They can be set at any desired angles to the reference axes by constructing $\boldsymbol{R}$ in (17) and (18) with the direction cosines of the required directions forming the columns of $\boldsymbol{R}$ in the required order. In any given ${}^{s}\boldsymbol{D}$ the eigenvalues specify the magnitudes of the deformation ellipsoid axes and the eigenvectors their orientations.

A deformation matrix ${}^{s}\boldsymbol{D} = \boldsymbol{R}(\boldsymbol{M}^{1/2})^{-1}\boldsymbol{R}^T$ can be set to operate at any desired attitude to the reference axes, specified by $\boldsymbol{R}_1$, as follows

$${}^{s}\boldsymbol{D} = \boldsymbol{R}_1\ \boldsymbol{R}^{T\,s}\boldsymbol{D}\boldsymbol{R}\ \boldsymbol{R}^T{}_1.$$

A deformation matrix ${}^{s}\boldsymbol{D}$ can be used to deform an ellipsoid

$$\boldsymbol{E}_R = {}^{s}\boldsymbol{D}^{-1}\boldsymbol{E}_0\,{}^{s}\boldsymbol{D}^{-1} = \boldsymbol{E}_1^{1/2}\ \boldsymbol{E}_0\boldsymbol{E}_1^{1/2}$$

where

$$\boldsymbol{E}^{1/2} = (\boldsymbol{R}\ \boldsymbol{M}\ \boldsymbol{R}^T)^{1/2} = \boldsymbol{R}\ \boldsymbol{M}^{1/2}\boldsymbol{R}^T.$$

When ${}^{s}\boldsymbol{D}_1$ is used to deform an ellipsoid, $\boldsymbol{E}_0$, into a resultant ellipsoid, $\boldsymbol{E}_R$, the axes of the deformed ellipsoid rotate continuously from its starting attitude to its resultant attitude. In general, it is only the ellipsoid axes of the imposed or incremental deformation ellipsoid, $\boldsymbol{E}_1$ which remain stationary during irrotational strain. In the special case in which the ellipsoid axes for both the original and the superposed deformations are parallel but not necessarily largest to largest etc. both sets of axes remain stationary during the deformation which is then called co-axial. If $\boldsymbol{E}_0 = \boldsymbol{R}\boldsymbol{M}_0\boldsymbol{R}^T$ and ${}^s\boldsymbol{D} = \boldsymbol{R}\boldsymbol{M}_1\boldsymbol{R}^T$ then

$$\boldsymbol{E}_R = \boldsymbol{R}\,\boldsymbol{M}^{\frac{1}{2}}{}_1\,\boldsymbol{R}^T\,\boldsymbol{R}\,\boldsymbol{M}_0\,\boldsymbol{R}^T\,\boldsymbol{R}\,\boldsymbol{M}^{\frac{1}{2}}{}_1\,\boldsymbol{R}^T = \boldsymbol{R}\,\boldsymbol{M}_R\,\boldsymbol{R}^T.$$

The deformation path for a co-axial deformation and the lack of rotation of the resultant ellipsoid axes during the deformation are shown in Fig. 1. If the original ellipsoid $\boldsymbol{E}_0$ is rotated so that its axes are no longer parallel to those of ${}^s\boldsymbol{D}$ and the same deformation superposed the resultant ellipsoid $\boldsymbol{E}_R$ rotates during the deformation (Fig. 1). The behaviour of the deformation ellipsoid itself during the deformation can be monitored by applying the same deformation to a sphere, $\boldsymbol{E}_0 = \boldsymbol{I}$. Figure 1 shows that the axes of the deformation ellipsoid remain stationary during the deformation.

Deformation matrices, ${}^s\boldsymbol{D}_1$, can be combined by multiplication but such multiplications are commutative only for coaxial deformation. In non-coaxial deformations deformation matrices must be multiplied together in the order in which the deformations occur, that is if the order is 1, 2, 3, then

$$\boldsymbol{l}_3 = {}^s\boldsymbol{D}_3{}^s\boldsymbol{D}_2{}^s\boldsymbol{D}_1\boldsymbol{l}_0 = {}^s\boldsymbol{D}\boldsymbol{l}_0.$$

In general det ${}^s\boldsymbol{D} = d \neq 1$ and the operation performed by ${}^s\boldsymbol{D}$ involves dilatation. It is convenient to define dilatation as a change in dimensions dependent only on volume change so that for $d<1$, X, Y, Z decrease in length or remain unaltered and for $d>1$, X, Y, Z either increase in length or remain unaltered. All other cases must involve dilatation according to these rules, together with a coincident tectonic non-dilatational deformation (Flinn 1978, p.298). Purely dilatational deformations are limited to those for which

for $d<1$, $X^2/YZ<1/d$; for $d>1$, $XY/Z<d$

and less precisely

for $d<1$, $X<d$ and for $d>1$. $Z<d$.

For coaxial deformation it is possible to combine a dilatational deformation matrix and a tectonic deformation matrix by multiplication to obtain a combined dilatational-tectonic deformation matrix since matrices commute in multiplication for coaxial strain. However, for non-coaxial deformation this is not true and whichever matrix premultiplies the other must be thought as a later superposed deformation. However, if the ellipsoids representing the incremental dilatational and incremental tectonic components are nearly coaxial then the error may be small as in Fig. 5 of Flinn (1978).

Only for coaxial deformations can a dilatation component and a tectonic component be combined by matrix multiplication to give a dilatation-tectonic (or non-constant volume) deformation. A dilatation-tectonic deformation can only be resolved into a dilatation and a tectonic component when the deformation is coaxial, and then there are an infinite number of pairs of dilatational and tectonic deformation ellipsoids which combine to form the required resultant. For instance, a spherical purely dilatational deformation, ${}^s\boldsymbol{D} = n\boldsymbol{I}$ where n is a positive real number, could be the result of a non-spherical dilatational deformation combined with a purely tectonic deformation. Nevertheless the dilatation ellipsoid is an essential concept in the study of dilatational-tectonic deformations. The restriction of dilatation to a pure compaction represented by a prolate spheroidal ellipsoid or to an isotropic volume change would be unnatural. It is unlikely that dilatation could take place in a tectonite without being influenced by its fabric.

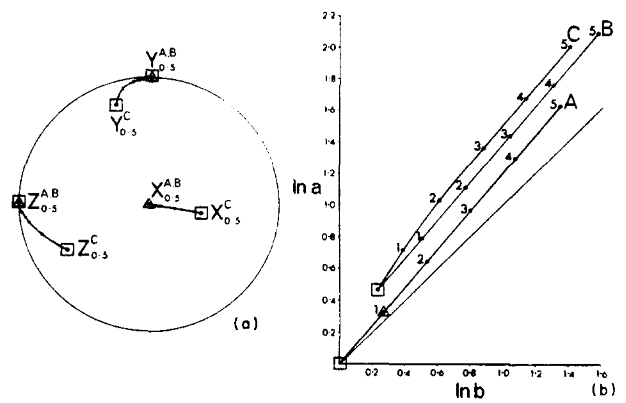

Fig. 1. (a) Equal-area projection. (b) Deformation plot. Coaxial deformation of a sphere $\boldsymbol{E}_0^A$. X^A, Y^A, Z^A in (a). Deformation path A in (b). Coaxial deformation of an ellipsoid $\boldsymbol{E}_0^B$. X^B, Y^B, Z^B in (a). Deformation path B in (b). Irrotational non-coaxial deformation of an ellipsoid $\boldsymbol{E}_0^C$. X^C, Y^C, Z^C in (a). Deformation path C in (b).

Starting ellipsoid — ⊡, $\boldsymbol{E}_0^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$\boldsymbol{E}_0^B = \begin{bmatrix} 0.38 & 0 & 0 \\ 0 & 1.21 & 0 \\ 0 & 0 & 2.17 \end{bmatrix}, \boldsymbol{E}_0^C = \begin{bmatrix} 0.84 & -0.13 & -0.72 \\ -0.13 & 1.38 & 0.36 \\ -0.72 & 0.36 & 1.55 \end{bmatrix}$$

Deformation matrix — ▲ — 5 increments,

$$\boldsymbol{D}^{A,B,C} = \begin{bmatrix} 1.47 & 0 & 0 \\ 0 & 0.98 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

N.B. In Figs. 1–5 the reference axes plot as $+\mathbf{x}_1$ — centre, $+\mathbf{x}_2$ — top, $+\mathbf{x}_3$ – r.h.s. of the projection, i.e. $\mathbf{x}_1$ is zenith, $\mathbf{x}_2$ is north and $\mathbf{x}_3$ is east.

NON-ORTHOGONAL NON-SYMMETRICAL DEFORMATION MATRICES, ${}^n\boldsymbol{D}$

There is no relationship which may be used to simplify and so to produce a unique, geologically significant, solution to equation (5) for this type of matrix. There is an infinite number of solutions for ${}^n\boldsymbol{D}$, for all of which the homogeneous strain resulting from the use of ${}^n\boldsymbol{D}$ as a transformation matrix is of the type called rotational because the axes of the deformation ellipsoid rotate during the operation. Several solutions will be considered.

Simple shear, ${}^{ss}\boldsymbol{D}$

$${}^{ss}\boldsymbol{D} = \boldsymbol{R}\,{}^{ss}\boldsymbol{C}\,\boldsymbol{R}^T \qquad (20)$$

$$\text{where } \left[{}^{ss}\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

here ${}^{ss}\boldsymbol{C}$ in (20) has been chosen as the canonical form. By a suitable choice of $\boldsymbol{R}$ the element 2^k can be rotated to any position occupied by 0 in (20), or it can be split to replace two zero elements in any column or in any row, or it can be distributed more widely. For all these cases the deformation caused by ${}^{ss}\boldsymbol{D}$ is still classical simple shear but in each case the shear elements are differently oriented with respect to the reference axes.

If the shear plane is oblique to all three reference axes there are no zeros in the deformation matrix and its simple shear nature is not immediately recognisable. However, its asymmetry makes obvious its rotational nature. The presence of zeros shows that the shear plane is parallel to one or two reference axes. Two off-diagonal zeros in the ith column indicate that the shear plane is parallel to the ith reference axis. In the canonical form, equation (20), there are two such columns indicating that the shear plane is parallel to the first and second (or $\mathbf{x}_1$ and $\mathbf{x}_2$) reference axes. The orientation of the shear direction is indicated by the row in which a non-zero element occurs. In the canonical form, equation (20), the non-zero element is in the first row so that the shear direction is parallel to the first (or $\mathbf{x}_1$) reference axis. Rotation of the simple shear system into a less symmetrical orientation relative to the reference axes reduces the number of zeros or changes their positions in the matrix as displayed in Fig. 2.

It is sufficient to study the operation of the canonical form

$$({}^{ss}\boldsymbol{C}^{-1})^T\,{}^{ss}\boldsymbol{C}^{-1} = \boldsymbol{E} = \begin{bmatrix} 1 & 0 & -2k \\ 0 & 1 & 0 \\ -2k & 0 & 4k^2+1 \end{bmatrix} = \boldsymbol{R}\,\boldsymbol{M}\,\boldsymbol{R}^T \quad (21)$$

$$\text{where } \boldsymbol{M} = \begin{bmatrix} ((k^2+1)^{\frac{1}{2}}+k)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & ((k^2+1)^{\frac{1}{2}}-k)^2 \end{bmatrix}$$

$$\text{and } \boldsymbol{R} = \begin{bmatrix} (((k^2+1)^{\frac{1}{2}}+k)^2+1)^{-\frac{1}{2}} & 0 & ((k^2+1)^{\frac{1}{2}}+k)\,(((k^2+1)^{\frac{1}{2}}+k)^2+1)^{-\frac{1}{2}} \\ 0 & 1 & 0 \\ -((k^2+1)^{\frac{1}{2}}+k)\,(((k^2+1)^{\frac{1}{2}}+k)^2+1)^{-\frac{1}{2}} & 0 & (((k^2+1)^{\frac{1}{2}}+k)^2+1)^{-\frac{1}{2}} \end{bmatrix}$$

The elements along the leading diagonal of $\boldsymbol{M}$ are the eigenvalues and the columns of $\boldsymbol{R}$ are the eigenvectors of ${}^{ss}\boldsymbol{C}$ obtained by the solution of its characteristic equation. From the eigenvalues the deformation ellipsoid axes may be obtained,

$$X,\ Y,\ Z = (k^2+1)^{\frac{1}{2}}+k,\ 1,\ (k^2+1)^{\frac{1}{2}}-k \quad (22)$$

and the orientations of the ellipsoid axes are given by the eigenvectors from which it may be seen that the X ellipsoid axis is inclined at α to $\mathbf{x}_3$ reference axis where

$$\alpha = \tan^{-1}((k^2+1)^{\frac{1}{2}}+k) = \tan^{-1}X. \quad (23)$$

These results can all be obtained algebraically (Jaeger 1956, p.33). Since $2k = (X^2-1)/X$ it is possible to express ${}^{ss}\boldsymbol{C}$ in terms of the major ellipsoid axis

$${}^{ss}\boldsymbol{C} = \begin{bmatrix} 1 & 0 & (X^2-1)/X \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

or the angle α through which that axis has been sheared

$${}^{ss}\boldsymbol{C} = \begin{bmatrix} 1 & 0 & (\tan^2\alpha-1)/\tan\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The ellipsoid axes and the eigenvectors obtained above can be substituted into (18) to obtain that symmetrical deformation matrix, ${}^{s}\boldsymbol{D}$, which produces a deformation ellipsoid identical to that produced by ${}^{ss}\boldsymbol{C}$. Using

$$\boldsymbol{R} = \begin{bmatrix} (X^2+1)^{-\frac{1}{2}} & 0 & X(X^2+1)^{-\frac{1}{2}} \\ 0 & 1 & 0 \\ -X(X^2+1)^{-\frac{1}{2}} & 0 & (X^2+1)^{-\frac{1}{2}} \end{bmatrix} \text{ and } \boldsymbol{M} = \begin{bmatrix} X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/X \end{bmatrix}$$

$${}^{s}\boldsymbol{D} = \begin{bmatrix} (X^3+1)/X(X^2+1) & 0 & (X^2-1)/(X^2+1) \\ 0 & 1 & 0 \\ (X^2-1)/(X^2+1) & 0 & 2X/(X^2+1) \end{bmatrix}. \quad (25)$$

Although the deformation matrices ${}^{s}\boldsymbol{D}$ from (25) and ${}^{ss}\boldsymbol{C}$ from (20) deform a unit sphere into identical deformation ellipsoids with identical orientations the deformations giving rise to those two ellipsoids are very different, one being irrotational and the other rotational, as may be seen by comparing ${}^{s}\boldsymbol{D}$ in (25) and ${}^{ss}\boldsymbol{C}$ in (24).

Progressive deformation by simple shear can be studied by treating the deformation as a series of increments, each incremental deformation being given by either equation (12), for lineations, or equation (11) for poles of planes respectively. If each increment of the deformation is identical to the preceding one then the sum of n increments is given by

$$\boldsymbol{E} = (\boldsymbol{D}^{-1})^T(\boldsymbol{D}^{-1})^T \ldots (\boldsymbol{D}^{-1})^T(\boldsymbol{D}^{-1}) \ldots (\boldsymbol{D}^{-1})(\boldsymbol{D}^{-1}) = [(\boldsymbol{D}^{-1})^T]^n[\boldsymbol{D}^{-1}]^n$$

$$\text{and } {}^{ss}\boldsymbol{C}^n = \begin{bmatrix} 1 & 0 & 2nk \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So that the substitution of nk for k in equations (21), (22) and (23) above gives the shapes and orientations of the deformation ellipsoids producing the deformation up to and including the nth stage.

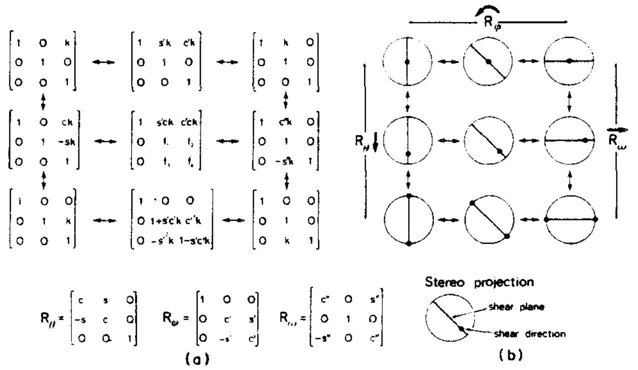

Fig. 2. Equivalent representations of simple shear. (a) Deformation matrices. (b) Diagrammatic stereo-plots. $f_1 = 1 - s\,s'\,c'\,k$; $f_2 = -s\,c'^2\,k$; $f_3 = s\,s'^2\,k$; $f_4 = 1 + c'\,s'\,s\,k$; $c = \cos\theta$, $s = \sin\theta$, $c' = \cos\phi$, $s' = \sin\phi$, $c'' = \cos\omega$, $s'' = \sin\omega$. R_θ, R_ϕ, R_ω – orthogonal matrices causing the rotations indicated in (b).

Figure 3 shows the deformation path and movement paths for the resultant deformation ellipsoid for five increments of simple shear deformation superimposed on a sphere. From them it is clear that the simple shear deformation matrix caused both distortion and rotation and thus can be said to rotate during the deformation. Figure 3 also shows the deformation paths and movement paths for simple shear deformations imposed on

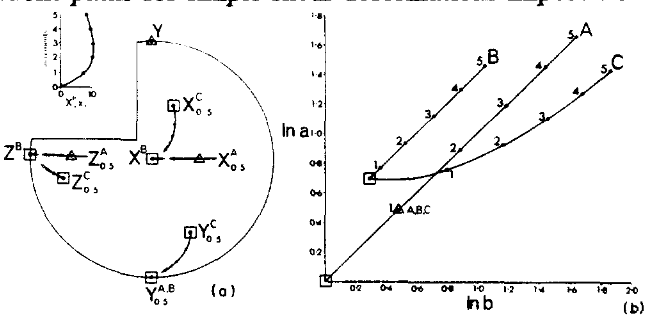

Fig. 3. (a) Equal-area projection. (b) Deformation matrix. Simple shear of a sphere E^A_0. X^A, Y^A, Z^A in (a). Deformation path A in (b). Simple shear of an ellipsoid E^B_0. X^B, Y^B, Z^B in (a) & inset. Deformation path B in (b). Simple shear of an ellipsoid E^C_0. X^C, Y^C, Z^C in (a). Deformation path C in (b). Starting ellipsoids as in Fig. 1. Deformation matrix – ▲ – 5 increments.

$$\boldsymbol{D}^{A,B,C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

ellipsoids with axes parallel and oblique to the reference axes at the start of the deformation.

Generalized simple shear, $^{gs}\mathbf{D}$

Simple shear is a very special type of deformation. It may be generalised by allowing the sliding 'layers' to deform during shearing. For example, if they extend by a factor f parallel to the rotation axis, by a factor e parallel to the sliding direction, and if the thickness changes by a factor g then the canonical form becomes

$$^{gs}C = \begin{bmatrix} e & 0 & 2k \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix} \qquad (26)$$

where e, f and g must all be positive for geological-type deformations. In general, $^{gs}\boldsymbol{D} = \boldsymbol{R}\,^{gs}\boldsymbol{C}\boldsymbol{R}^T$. The dilation for such a deformation is given by $d = \det\,^{gs}\boldsymbol{C} = efg$.

The deformation ellipsoid is given by (27)

$$\boldsymbol{E} = \boldsymbol{R}(^{gs}\boldsymbol{C}^{-1})^T\,^{gs}\boldsymbol{C}^{-1}\boldsymbol{R}^T = \boldsymbol{R}\begin{bmatrix} e^{-2} & 0 & -d/e^2g \\ 0 & f^{-2} & 0 \\ -d/eg^2 & 0 & (d^2+e)^2/e^2g^2 \end{bmatrix}\boldsymbol{R}^T$$

Solution of the characteristic equation of the central matrix on the right of (27) gives the eigenvalues as

$$f^{-2},\ ((g^2+e^2+d^2) \pm (g^2+e^2+d^2)^2 - 4e^2g^2)/2e^2g^2,$$

the ellipsoid axes being the reciprocals of the square roots of the eigenvalues. The orientations of the ellipsoid axes are given by the eigenvectors. However, there seems little point in obtaining all these algebraic solutions since they show no signs of simplifying. Numerical solutions are easily obtained by computer methods directly from the matrices without any algebraic expansions. As with simple shear, deformation paths and movement paths are obtainable by direct numerical solution of the simple matrix equations (13), (12) and (11), substituting matrix (26) for D.

Figure 4 shows the drastic changes in deformation paths and movement paths which arise from the combination of pure shear in this way and with the maximum, intermediate and minimum axes of the pure shear component taking up to six possible non-oblique orientations relative to the simple shear system. Deformations of this type are probably of considerable geological significance, especially in shear zones. Exten-

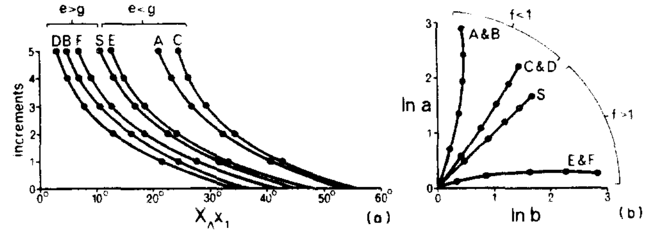

Fig. 4. Rotational deformation: generalised simple shear of a sphere

$$\boldsymbol{E}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Deformation matrix } = \begin{bmatrix} e & 0 & 1 \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix} \text{— 5 increments.}$$

For $e = f = g = 1$, see deformation path S in (b) and graph S in (a) — identical to A in Fig. 1(b) and X^A, Y^A, Z^A in Fig. 1(a). For $X = 1.24$, $Y = 0.95$, $Z = 0.85$: (A) $e = Y, f = Z, g = X$; (B) $e = X, f = Z, g = Y$; (C) $e = Z, f = Y, g = X$; (D) $e = X, f = Y, g = Z$; (E) $e = Z, f = X, g = Y$; (F) $e = Y, f = X, g = Z$.

sions parallel to the simple shear rotation axes gives rise to *s*-tectonites while shortening in this direction gives rise to *l*-tectonites.

Ramberg (1976) and Ramberg & Ghosh (1977) have studied the movement paths of structural directions during deformations of this type for the special case of zero dilatation ($d = abc = 1$) and making use of rates of deformation instead of increments.

The canonical form (26) above is not that for completely generalised simple shear because the deformation of the sliding 'layers' takes place with a special orientation relative to the shear movements.

The completely generalised simple shear deformation matrix is given by

$$^{gs}\boldsymbol{D}' = \boldsymbol{R}'\,^{gs}\boldsymbol{D}\boldsymbol{R}'^T \qquad (28)$$

$$\left[\text{where } ^{gs}\boldsymbol{D} = \begin{bmatrix} ea_{11} & a_{12} & a_{13} \\ a_{21} & fa_{22} & a_{23} \\ a_{31} & a_{32} & ga_{33} \end{bmatrix}\right]$$

$$\text{and } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \boldsymbol{R}\begin{bmatrix} 1 & 0 & 2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{R}^T.$$

Thus, $^{gs}\boldsymbol{D}'$ in (28) causes a deformation of simple shear type at any orientation, $\mathbf{R}$, to the reference axes combined with a coincident pure shear specified by factors (ellipsoid axes) e, f, and g at any orientation, $\boldsymbol{R}'$, to the reference axes. The algebraic investigation of this type of deformation is even less rewarding than it is for the partially generalized form mentioned above. However, for any numerically specified case $^{gs}\boldsymbol{D}'$ can be constructed with the aid of (28) and the deformation paths and movement paths found as easily as for the simplest deformations by use of the simple matrix equations (13), (11) and (12).

Other types of rotational deformation

Deformation matrices for other types of rotational deformation may be constructed by much the same method that was used above for generalised simple shear, and also in other ways.

For instance, the purely rotational strain caused by an orthogonal deformation matrix, $^o\boldsymbol{D}$, may be made distortional and rotational as follows:

$$\left[\ {}^{sg}\boldsymbol{D} = \boldsymbol{R}\begin{bmatrix} e & 0 & 0 \\ 0 & f\cos\theta & \sin\theta \\ 0 & -\sin\theta & g\cos\theta \end{bmatrix}\boldsymbol{R}^T\ \right]$$

For the completely general case where

$${}^{c}\boldsymbol{D} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \boldsymbol{R}'\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}\boldsymbol{R}'^T$$

$${}^{sg}\boldsymbol{D} = \boldsymbol{R}\begin{bmatrix} el_1 & l_2 & l_3 \\ m_1 & fm_2 & m_3 \\ n_1 & n_2 & gn_3 \end{bmatrix}\boldsymbol{R}^T. \qquad (29)$$

For geologically significant deformations, e, f and g must be positive. The geometrical interpretation of the deformation caused by (29) parallels that given above for (28).

Irrotational strain caused by the symmetrical deformation matrix, ${}^{s}\boldsymbol{D}$, can be made rotational as follows:

$${}^{g}\boldsymbol{D} = \boldsymbol{R}_2\ {}^{s}\boldsymbol{D}\ \boldsymbol{R}_1^T.$$

Further rotational-type deformation matrices may be constructed by combining different simple shear matrices. The deformation caused by such matrices may be called compound shear. Such deformation can be thought of as resulting from the combination of different systems of simple shear acting at the same time and combining to produce a non-simple shear-type deformation. However, combination of two simple shear systems with the shear planes normal to each other and with both shear directions parallel to the line of intersection of the planes does not give rise to compound shear, merely to a single resultant rotated simple shear system.

For example

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} 1 & k' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & k' & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which can be seen from Fig. 2 to be simple shear with the shear plane oblique to two reference axes and the shear direction parallel to the other. If two simple shear systems are combined so that the shear planes are parallel but the shear directions are not, the resultant is again a simple system (Fig 2);

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix}.$$

If two simple shear systems are combined so that both shear directions are normal to the line of intersection of the shear planes then the resultant matrix is either symmetrical, and therefore produces irrotational strain, or asymmetrical and produces generalised simple shear;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k' & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & k' & 1 \end{bmatrix} \text{similar to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & f & k'' \\ 0 & 0 & g \end{bmatrix}.$$

On the other hand if the two shear planes intersect and no more than one shear direction is normal or parallel to the intersection then the resultant is compound shear. In the simplest case

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k' & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & k' & 1 \end{bmatrix}$$

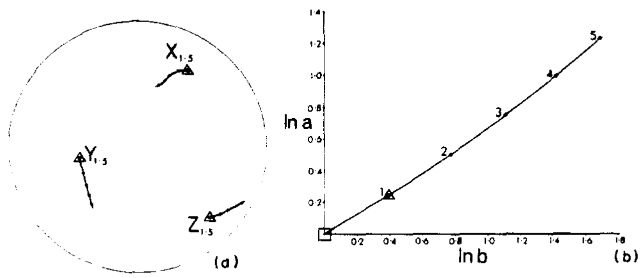

Fig. 5. (a) Equal-area projection. (b) Deformation matrix. Rotational deformation: compound shear of a sphere,
$\boldsymbol{E}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Deformation matrix, $\boldsymbol{D} = \begin{bmatrix} 1 & 0.4 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$ — five increments.

$$\text{or} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix} \text{similar to} \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & k' & 1 \end{bmatrix}.$$

With both these deformation matrices, directions parallel to the $\boldsymbol{X}$ axis remain unaltered in length and orientation. All other directions change in length and orientation (Fig. 5). This type of deformation matrix can be generalised in much the same way that the simple shear deformations matrix was generalised above.

Three simple shear systems may be combined orthogonally to give matrices similar to

$$\begin{bmatrix} 1 & e & 0 \\ 0 & 1 & f \\ g & 0 & 1 \end{bmatrix}.$$

These cause deformations involving a volume change; $\det{}^{c}\boldsymbol{D} \neq 1$. It is apparent that there is no limit to the number of different types of deformation matrix which may be constructed. Algebraic investigation of the nature of these deformations, such as the determination of the axes and their orientation for the deformation ellipsoid and the equations defining the deformation paths and the movement paths of structural directions by the expansion of the simple matrix equations, serves little purpose. The algebraic equations are not usually amenable to simplification and reveal less about the nature of the deformation than can be gained from a study of the canonical form of the deformation matrix and the numerical solution of the matrix equations.

Few if any of the types of rotational deformation mentioned above are immediately recognisable as of geological interest, with the exception of simple shear. It seems unlikely that simple shear is the only or even the dominant type of rotational rock deformation. This work reveals the multifarious nature of rotational deformation and the need to identify those types which occur naturally. It also shows that deformations of different type (i.e. not similar) can be combined as easily as they can be applied sequentially.

DISCUSSION

It is possible to construct deformation matrices to produce innumerable different deformations, rotational and otherwise. The shape, orientation and volume of the deformation ellipsoid can be found for any deformation matrix with the aid of equation (13), but a deformation ellipsoid is not unique to any deformation matrix and cannot be used to characterise or represent the deformation unambiguously. For that, the deformation matrix is required. If the deformation matrix is symmetrical or orthogonal the nature of the deformation it causes is obvious and may be quantified easily by means described in the appropriate sections above. However, it is much more difficult to determine the nature of the deformation caused by any given deformation matrix which is neither orthogonal nor symmetrical. Even a systematic study of the patterns of the movement paths obtained from its operation, and a determination of its deformation ellipsoid may not suffice to classify it sufficiently precisely or allocate it to its canonical form. A powerful method which may be used for the investigation of such matrices is to attempt to reduce them to upper triangular form by means of the *Q–R* algorithm (Hammarling 1970, pp. 138–46).

The deformation matrices for some of the more complex types of deformation cannot be reduced to real upper triangular forms. However, under the action of the *Q–R* algorithm matrices similar to

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} e & 0 & k \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} e & 0 & 0 \\ 0 & f\cos\theta & \sin\theta \\ 0 & -\sin\theta & g\cos\theta \end{bmatrix},$$

$$\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & k' \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} e & k & 0 \\ 0 & f & k' \\ 0 & 0 & g \end{bmatrix}$$

converge on the canonical forms, often very rapidly. However, it is necessary to monitor the convergence as in some cases, after an initial close approach, the convergence becomes a divergence.

Finally, it should be noted that although this paper has been concerned solely with investigating the special case of homogeneous strain operating without translation, some of the methods developed may be suitable for use in the study of more generalised forms of deformation. If the elements in the deformation matrix are replaced by suitable functions the deformation caused becomes inhomogeneous strain. Furthermore, if the deformation matrix is made into a 4 × 4 matrix it can perform operations involving translation as well as deformation.

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